Lecture 8 Cubic and Quartic equations

Part I (Theory)

Start with Newton's relation between roots and coefficients:

$$x^n - c_1 x^{n-1} + \cdots + (-1)^n c_n = (x - x_1) \cdots (x - x_n)$$

It follows that

$$c_1 = x_1 + x_2 + \cdots + x_n$$

$$c_2 = x_1 x_2 + x_2 x_3 + \cdots + x_n x_1$$

$$\vdots$$

$$c_n = x_1 x_2 \cdots x_n$$

$\{c_1, c_2, \ldots, c_n\}$ elementary symmetric polynomials.

Now, consider the group action:

$$\begin{array}{ccc} S_n \times \mathbb{K}[x_1, \ldots, x_n] & \longrightarrow & \mathbb{K}[x_1, \ldots, x_n] \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (\sigma, f) & \longmapsto & \sigma(f) \end{array}$$

where

$$f = \Sigma\, a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n}$$

$$\sigma(f) = \Sigma\, a_{i_1\dots i_n} x_{\sigma(1)}^{i_1} \cdots x_{\sigma(n)}^{i_n}$$

Example: $\quad f = x_1^2 + x_2 x_3 \in K[x_1, x_2, x_3]$

$$\sigma = [1\ 2\ 3] \in S_3$$

$$\sigma(f) = x_2^2 + x_3 x_1$$

Note that, $\forall\, c_i$, elementary sym. poly.

$$\begin{array}{c} c_i \\ \| \\ c_i(x_1, \dots x_n) \end{array}$$

$\forall\, \sigma \in S_n, \quad \sigma(c_i) = c_i.$ That is why we call c_i symmetric poly.

Def: $f \in k[x_1, \dots x_n]$ is called symmetric, if

$S_n(f) = f.$ (i.e. f is ~~the~~ fixed pt under S_n-action).

Theorem (Gauß).

Any symmetric polynomial $f \in k[x_1, \dots, x_n]$ is a polynomial of $c_1, \dots, c_n$.

Explanation of Gauß's Theorem:

Look at the orbit of f under the group action:

$$
\begin{array}{llll}
O_f = \{ & f, & [12](f), & [13](f) \} \\
 & \| & \| & \| \\
 & x_1^2 + x_2x_3 & x_2^2 + x_1x_3 & x_3^2 + x_2x_1 \\
 & \| & \| & \| \\
 & [23](f) & [123](f) & [132](f)
\end{array}
$$

Note: $\forall \sigma \in S_3,\ \sigma\big(f + [12](f) + [13](f)\big)$

$$= f + [12](f) + [13]f \ !$$

In fact, $f + [12](f) + [13](f) = (x_1^2 + x_2^2 + x_3^2) + (x_1x_2 + x_2x_3 + x_3x_1)$

$$= (x_1 + x_2 + x_3)^2 - (x_1x_2 + x_2x_3 + x_3x_1)$$

$$= c_1^2 - c_2.$$

Def (field of rational functions)

$$k(x_1, \dots, x_n) = \left\{ \frac{f}{g} \;\middle|\; f, g \in k[x_1, \dots, x_n],\ g \neq 0 \right\}$$

$$\frac{f_1}{g_1} + \frac{f_2}{g_2} = \frac{f_1 g_2 + f_2 g_1}{g_1 g_2}$$

$$\frac{f_1}{g_1} \cdot \frac{f_2}{g_2} = \frac{f_1 \cdot f_2}{g_1 \cdot g_2}$$

We have the natural extension of group action:

$$k[X_1, \dots, X_n] \subset k(X_1, \dots, X_n)$$

$$f \longmapsto \frac{f}{1}$$

$$S_n \times k(X_1, \dots, X_n) \longrightarrow k(X_1, \dots, X_n)$$

$$\left(\sigma, \frac{f}{g}\right) \longmapsto \sigma\left(\frac{f}{g}\right) \stackrel{\Delta}{=} \frac{\sigma(f)}{\sigma(g)}$$

Cor of Gauß's Theorem:

If $\phi = \frac{f}{g}$ is fixed by S_n-action (i.e. symmetric rational fn)

then it is a rational fn of $c_1, \dots, c_n$.

Theorem (Lagrange)

Let $\phi, \psi \in k(x_1, \dots, x_n)$ and $G = S_n$.

If one has $\quad$ ($\Leftrightarrow$ $G_{\psi}(\phi) = \phi$)

$$G_\psi \leq G_\phi ,$$

then ϕ is a rational fn of $\psi, c_1, \dots, c_n$.

Pf: write

$$G \cdot \psi = \{ \overset{\psi}{\overset{\shortparallel}{\psi_1}}, \dots, \psi_s \}$$

to be the orbit under G-action.

Then $\psi_i = \sigma_i(\psi)$, for some $\sigma_i \in G$.

Set $\phi_i = \sigma_i(\phi)$, $\quad i = 1, \dots s$

Note: $G_\psi \leq G_\phi \Rightarrow \phi_1 = \phi$.

But ϕ_i might be equal to ϕ_j, $i \neq j$.

Consider $f(t) = (t - \psi_1) \cdots (t - \psi_s)$

$$g(t) = \cancel{\tfrac{1}{\ }} f(t) \cdot \left(\frac{\phi_1}{t - \psi_1} + \cdots + \frac{\phi_s}{t - \psi_s} \right)$$

Note : the coefficients of $f(t)$ and $g(t)$

are all symmetric rational functions !

By Cor to Gauss's theorem, the coeffs of f and g are

rational fns of $c_1, \ldots, c_n$.

Now put $t = \psi_1$ in $g(t)$, we get

$$g(\psi) = g(\psi_1) = f'(\psi_1)\cdot\phi_1 + 0 \cdots + 0 = f'(\psi)\,\phi$$

$$\Rightarrow \phi = \frac{g(\psi)}{f'(\psi)}$$ which is a rational fn of

ψ and $c_1, \ldots, c_n$.

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Cor 1: Suppose $V \in k[x_1, \ldots x_n)$ with $G_V = \{1\}$.

Then x_i, $i = 1, \ldots, n$ are rational fns of

V and $c_1, \ldots, c_n$.

Cor 2: Suppose for $\phi, \psi \in k[x_1, \ldots, x_n)$, $[G_\phi : G_\psi] = \nu \geq 1$,

Then $\exists\, F(t) \in k(x_1, \ldots x_n)[t]$, s.t

(i) $F(\psi) = 0$

(ii) The coefficients of $F(t)$ are rational fns of $\phi, c_1, \dots, c_n$.

pf: Write

$$G_\phi \cdot \psi = \{\overset{\substack{\psi \\ \parallel}}{\psi_1}, \dots, \psi_\nu\}$$

to be the G_ϕ-orbit of ψ.

Consider

$$F(t) = (t-\psi_1) \cdots (t-\psi_\nu) \in k(x_1, \dots, x_n)[t].$$

The coeff. of $F(t)$ are inv under G_ϕ-action.

By Lagrange's Theorem, they are rational fns of ϕ and $c_1, \dots, c_n$, as claimed.

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Proof of Gauss's theorem:

Let f be a symmetric poly. Write

$$f = f_0 + f_1 + \dots + f_d,$$ such that

each f_i is homogeneous of total degree i.

Then

$$\begin{array}{ccccc} \sigma(f) & = & \sigma(f_0) + \cdots + \sigma(f_d) & , & \forall \sigma \in S_n \\ \| & & \| \qquad\qquad \| & & \\ f & = & f_0 + \cdots + f_d & & \end{array}$$

$\Rightarrow$ f_i is sym. $\forall\ 0 \le i \le d$.

Thus we assume in the following that

f is homogenous of total degree d.

i.e. $f = \sum a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$

$$i_1 + \cdots + i_n = d$$

Now we put the dictionary order on $\{x_1, \cdots, x_n\}$:

Two monomials of degree d:

$$x^I = x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n} = x^J,$$

$|I| = |J| = d$, $|I| = \sum i_\ell$, $|J| = \sum j_\ell$

$\Longleftrightarrow \exists\, s \ge 1,\ i_1 = j_1, \cdots, i_{s-1} = j_{s-1},\ i_s > j_s$.

Under the dictionary order, the greatest monomial in

$$\sigma_1^{i_1} \cdots \sigma_n^{i_n} \text{ is } x_1^{i_1+i_2+\cdots+i_n} x_2^{i_2+\cdots+i_n} \cdots x_n^{i_n}$$

Since f is symmetric, the greatest monomial in f is of form

$$x_1^{\lambda_1} \cdots x_n^{\lambda_n}, \quad \text{with } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

Then $c_1^{\lambda_1-\lambda_2} c_2^{\lambda_2-\lambda_3} \cdots c_{n-1}^{\lambda_{n-1}-\lambda_n} c_n^{\lambda_n}$ has the same greatest monomial.

Thus $\tilde{f} = f - a_{\lambda_1 \cdots \lambda_n} c_1^{\lambda_1-\lambda_2} \cdots c_n^{\lambda_n}$ has the greatest monomial strictly less than $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$. Then we apply the induction on the order, to complete the proof on the existence.

Now, suppose $f \in k[X_1 \ldots X_n]$ with $f(c_1 \ldots c_n) = 0$. We show that $f = 0$. For this, we choose the dictionary order by $x_n > x_{n-1} \cdots > x_1$. Then, it is clear that the greatest monomial in $f(x_1, \ldots x_n)$ and the greatest monomial in $f(c_1, \ldots, c_n)$ corresponds to each other. Thus, it follows that $f = 0$.

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Part II (Application)

1. Solve

$$x^3 - c_1 x^2 + c_2 \cdot x - c_3 = 0$$

Step 1: $S_3 \ : \ c_1, c_2, c_3$

$\quad 2 \ \Big|$

$A_3 \ : \ \Delta = (x_1 - x_2)(x_2 - x_3)(x_1 - x_3)$

Note $G_\Delta = A_3 \triangleleft S_3$.

By Cor 2, Δ satisfies a degree 2 eqn with coefficients rational fns of c_1, c_2, c_3.

Indeed,

$$\Delta^2 = \underbrace{c_1^2 c_2^2 + 18 c_1 c_2 c_3 - 4 c_2^3 - 4 c_1^3 c_3 - 27 c_3^2}_{= (*)}$$

i.e Δ is a square root of $(*)$.

Step 2: $A_3 \ : \ \Delta \qquad \omega = \frac{-1+\sqrt{-3}}{2}$

$\quad 3 \ \Big|$

$\{1\} \ : \ V = x_1 + \omega x_2 + \omega^2 x_3$

Note: $V = V_1 = x_1 + \omega x_2 + \omega^2 x_3$

$[123]\,V = V_2 = x_2 + \omega x_3 + \omega^2 x_1 = \omega^2 V_1$

$[132]\,V = V_3 = x_3 + \omega x_1 + \omega^2 x_2 = \omega^4 V_1$

$[23]\,V = V_4 = x_1 + \omega \cdot x_3 + \omega^2 x_2$

$[13]\,V = V_5 = x_3 + \omega x_2 + \omega^2 x_1 = \omega^2 V_4$

$[12]\,V = V_6 = x_2 + \omega x_1 + \omega^2 x_3 = \omega^4 V_4$

Thus $G_V = \{1\}$.

Cor 2 $\Rightarrow$ V_i $(i = 1 \ldots 6)$ satisfies a degree 3 eqn with rational fns of Δ, c_1, c_2, c_3 as coeff.

Indeed,

$$\left.\begin{aligned} V_1^3 + V_4^3 &= 2c_1^3 - 9c_1c_2 + 27c_3 \\ V_1^3 - V_4^3 &= -3\sqrt{-3}\,\Delta \end{aligned}\right\} \Rightarrow$$

$$V^3 = \underbrace{\tfrac{1}{2}\left(2c_1^3 - 9c_1c_2 + 27c_3 - 3\sqrt{-3}\,\Delta\right)}_{(**)}$$

$\Rightarrow V = \sqrt[3]{**}$

Similarly:

$$V_4 = \sqrt[3]{\tfrac{1}{2}(2c_1^3 - 9c_1c_2 + 27c_3 + 3\sqrt{3}\,\Delta}$$

But note

$$V_1 V_4 = x_1^2 + x_2^2 + x_3^2 - (x_1x_2 + x_2x_3 + x_3x_1)$$

$$= c_1^2 - 3c_2$$

$$\Rightarrow V_4 = \frac{c_1^2 - 3c_2}{V}$$

step 3. By cor 1, x_i, $i = 1, 2, 3$ are rat. fn of

V, c_1, c_2, c_3.

Indeed.

$$x_1 + x_2 + x_3 = c_1$$

$$x_1 + \omega x_2 + \omega^2 x_3 = V$$

$$x_1 + \omega^2 x_2 + \underset{\omega}{\underset{\shortparallel}{\omega^4}} x_3 = V_4 = \frac{c_1^2 - 3c_2}{V}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix} \begin{pmatrix} c_1 \\ V \\ \frac{c_1^2 - 3c_2}{V} \end{pmatrix} \Rightarrow x_i = \cdots$$

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Summary:

$S_3 \;:\; c_1, c_2, c_3$

$|$

$A_3 \;:\; \Delta$

$|$

$\{1\} \;:\; V \rightsquigarrow \{x_1, x_2, x_3\}.$

2. Solve

$$x^4 - c_1 x^3 + c_2 x^2 - c_3 x + c_4 = 0$$

Outline:

$S_4 \;:\; c_1, c_2, c_3, c_4$

$3 \,|$

$G_8 \;:\; y = x_1 x_2 + x_3 x_4$

$2 \,|$

$G_4 \;:\; t = x_1 + x_2 - x_3 - x_4$

$2 \,|$

$G_2 \;:\; V^2$

$\{x_1, x_2, x_3, x_4\}.$

$2 \,|$

$G_1 \;:\; V = x_1 + i x_2 + i^2 x_3 + i^3 x_4 = x_1 + i x_2 - x_3 - i x_4$